From the last theorem we see that a regular group cannot be primitive unless it is generated by a single cycle involving a prime number of letters. Since such a group must be primitive we have the following

THEOREM : *The regular primitive groups and the prime numbers have a* 1,1 *correspondence ; i. e. for each prime number there is one regular primitive group and for each regular primitive group there is one prime number.*

[To be Continued.]

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Continued from February Number.]

II. VOLUMES. Let the density vary as $x^{h-1}y^{k-1}z^{l-1}$. Then

$$\bar{x}=\frac{\int\int\int x^h y^{k-1} z^{l-1} dxdydz}{\int\int\int x^{h-1} y^{k-1} z^{l-1} dxdydz},\quad \bar{y}=\frac{\int\int\int x^{h-1} y^k z^{l-1} dxdydz}{\int\int\int x^{h-1} y^{k-1} z^{l-1} dxdydz},$$

$$\bar{z}=\frac{\int\int\int x^{h-1} y^{k-1} z^l dxdydz}{\int\int\int x^{h-1} y^{k-1} z^{l-1} dxdydz}.$$

$$\therefore \bar{x}=\frac{\dfrac{a^{h+1}b^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\ \dfrac{\Gamma\left\{\frac{h+1}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n+1)\right\}\Gamma\left\{\frac{l}{2})2p+1)\right\}}{\Gamma\left\{\frac{h+1}{2}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}{\dfrac{a^hb^kc^l}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\ \dfrac{\Gamma\left\{\frac{h}{2}(2m+1)\right\}\Gamma\left\{\frac{k}{2}(2n=1)\right\}\Gamma\left\{\frac{l}{2}(2p+1)\right\}}{\Gamma\left\{\frac{h}{3}(2m+1)+\frac{k}{2}(2n+1)+\frac{l}{2}(2p+1)+1\right\}}}$$

$$\therefore\ \bar{x}=\frac{\Gamma(hm+m+\frac{h+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(hm+\frac{h}{2})\Gamma(hm+kn+lp+m+\frac{h+k+l+1}{2}+1)}a\ \dots\dots\dots (C).$$

$$\bar{y}=\frac{\Gamma(kn+n+\frac{k+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(kn+\frac{k}{2})\Gamma(hm+kn+lp+n+\frac{h+k+l+1}{2}+1)}b\ \dots\dots\dots\dots (D).$$

$$\bar{z}=\frac{\Gamma(lp+p+\frac{l+1}{2})\Gamma(hm+kn+lp+\frac{h+k+l}{2}+1)}{\Gamma(lp+\frac{l}{2})\Gamma(hm+kn+lp+p+\frac{h+k+l+1}{2}+1)}c\dots\dots\dots\dots (E).$$

This gives the centroid of the eighth part of the volume whatever may be the values of h, k, l, m, n, p.

Let $m=n=p$, and also let the density vary as xyz so that $h=k=l=2$.

$$\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(6m+4)}{\Gamma(2m+1)\Gamma(7m+\frac{9}{2})}.$$

Let $m=0$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(1)\Gamma(\frac{9}{2})}=\frac{16}{35}$, for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$.

Let $m=1$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(\frac{9}{2})\Gamma(10)}{\Gamma(3)\Gamma(\frac{23}{2})}=\frac{2^{13}}{11.13.17.19}$,

for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$.

Let $m=\frac{3}{2}$, $\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(6)\Gamma(13)}{\Gamma(4)\Gamma(15)}=\frac{10}{91}$,

for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$,

the centroid of the volume bounded by the positive portion of the co-ordinate planes.

Let $m=n=p$, and let the density be the same throughout the solid so that, $h=k=l=1$

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(2m+1)\Gamma(3m+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(4m+3)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)} = \frac{3}{8}$, for $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)} = \frac{21}{128}$, for $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$

Let $m=\frac{3}{2}$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c} = \frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)} = \frac{3}{28}$, for $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1.$

Let $m=n=p$, and let the density vary as xy, so that $h=k=2$, $l=1$,

$$\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(3m+\frac{3}{2})\Gamma(5m+\frac{7}{2})}{\Gamma(2m+1)\Gamma(6m+4)}, \quad \frac{\bar{z}}{c} = \frac{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}{\Gamma(m+\frac{1}{2})\Gamma(6m+4)}.$$

Let $m=0$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{\Gamma(1)\Gamma(4)} = \frac{5\pi}{32}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(1)\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})\Gamma(4)} = \frac{5}{16}, \text{ for} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Let $m=1$, $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(\frac{9}{2})\Gamma(\frac{17}{2})}{\Gamma(3)\Gamma(10)} = \frac{5.7.11.13.15\pi}{2^{20}}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(3)\Gamma(\frac{17}{2})}{\Gamma(\frac{3}{2})\Gamma(10)} = \frac{5.11.13}{2^{13}}, \text{ for} \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$$

Let $m=\frac{3}{2}$,— $\therefore \frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\Gamma(6)\Gamma(11)}{\Gamma(4)\Gamma(13)} = \frac{5}{33}$,

$$\frac{\bar{z}}{c} = \frac{\Gamma(4)\Gamma(11)}{\Gamma(2)\Gamma(13)} = \frac{1}{22}, \text{ for} \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1.$$

Let $m=n=p$, and let the density vary as x so that $h=2$, $k=l=1$.

$$\therefore \frac{\bar{x}}{a}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(4m+3)}{\Gamma(2m+1)\Gamma(5m+\frac{7}{2})}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(2m+1)\Gamma(4m+3)}{\Gamma(m+\frac{1}{2})\Gamma(5m+\frac{7}{2})}.$$

Let $m=0$, then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}=\frac{8}{15}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}=\frac{16}{15\pi}.$$

Let $m=1$, then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}=\frac{2^7}{3.11.13}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(3)\Gamma(7)}{\Gamma(\frac{3}{2})\Gamma(\frac{17}{2})}=\frac{2^{14}}{5.7.9.11.13\pi}.$$

Let $m=\frac{3}{2}$, then for $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$

$$\frac{\bar{x}}{a}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}=\frac{2}{9}, \quad \frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{\Gamma(4)\Gamma(9)}{\Gamma(2)\Gamma(11)}=\frac{1}{15}.$$

Thus we could multiply examples almost without number.

If $a=b$ we get another series of areas.

If $a=b=c$ we get another series of solids.

If $b=c$ or $a=c$ we get still another series of solids.

But formulæ (A), (B), (C), (D), (E) apply to them all.

One more example and we will proceed to the discussion of surfaces. Let the density vary as x^3y^2z, and let the equation to the surface be

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1,$$

so that $h=4$, $k=3$, $l=2$, $m=1$, $n=2$, $p=3$

$$\therefore \bar{x}=\frac{\Gamma(\frac{15}{2})\Gamma(\frac{43}{2})}{\Gamma(6)\Gamma(23)}a=\frac{5.7.9.11.13.23.29.31.37.39.41\pi a}{2^{50}}$$

$$\bar{y}=\frac{\Gamma(10)\Gamma(\frac{43}{2})}{\Gamma(\frac{15}{2})\Gamma(24)}b=\frac{5.9.29.31.37.41b}{11.2^{26}},$$

$$\bar{z}=\frac{\Gamma(\frac{21}{2})\Gamma(\frac{43}{2})}{\Gamma(7)\Gamma(25)}c=\frac{5.7.9.11.13.13.17.19.29.31.37.41\pi c}{2^{57}}.$$

The prodigious amount of work to accomplish this by the ordinary method would be impossible.

[To be Continued.]